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COMMENT

Motion in a random gyrotropic environment in D dimensions

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Abstract. We construct a simple model of ballistic motion in a random environment in an arbitrary number of dimensions. The motion is determined by a quenched set of random orthogonal matrices on a hypercubic lattice, relating incoming and outgoing directions at a particular site. We select, study and classify a set of these matrices which agree with intuitive notions of 'isotropy'. A mean field theory is constructed of the transition between localised and extended trajectories and a qualitative discussion of various features of the phase diagram is made.

Ehrenfest wind-tree and Lorentz gas models are the simplest interesting models of ballistic motion in a random environment (e.g. Spohn 1980, Gates and Westcott 1982). Here we wish to consider, in arbitrary dimensions, models with more general relations between incoming and outgoing velocities (at a scattering event) than merely reflections at an obstacle. In so doing we extend previous work for the two-dimensional case (Gunn and Ortuño 1985). In this comment we restrict ourselves to classifying the models in higher dimensions, constructing a mean field theory and giving some qualitative arguments about the 'phase diagram' of regions of localised and extended trajectories.

The models consist of a particle moving on a d-dimensional cubic lattice, its motion being determined by 'gyroscopic instructions' residing at each site. More specifically, the particle moves between nearest-neighbour sites on the lattice, with the direction of motion at a given time step being determined by the direction at the previous time step in a one-to-one manner via a quenched random matrix associated with the site. The precise nature of the matrices will be considered below, but they are generalisations of the rotation matrices of the 2D case. A particular realisation of the model is characterised by the set of probabilities, $P(M)$, that if a site is chosen at random, it will have a particular matrix, say M, associated with it.

We wish to study the statistical nature of the trajectories in such models. It is convenient to note at this point that the one-to-one nature of the matrices and the fact that they are quenched imply that trajectories cannot branch, merge or end: any trajectory must be either infinite or cyclic.

We now turn to the consideration of the nature of the random matrices, primarily guided by comparison with the two-dimensional case (for instance, the angle between incident and departing directions, at a given site, should be the same in all cases). Let us now discuss the symmetries and other characteristics required of the matrices.

(i) A key feature of the two-dimensional model, which will be retained in higher dimensions, was that the oppositely directed incoming trajectories emerged in opposite directions. Thus we need only consider $d \times d$ rather than $2d \times 2d$ matrices. We call this 'reflection symmetry'.

(ii) The one-to-one nature of the relation between incoming and outgoing directions implies that the matrices must be orthogonal. To see this, note that one to one implies that there is only one entry in any row or column of a matrix. Thus to construct M^+M to test for orthogonality, the only non-zero contribution will be when a row in the transpose multiplies the equivalent column in the matrix, i.e. the diagonal elements. Since the outgoing vectors are of the same length as the incoming ones and normalised to unity, $M^+M = I$.

(iii) We want our transformations to be 'isotropic'. One could expect that this would result in the matrices commuting with all rotations of $\pi/2$ around the axes. However, one can prove that any matrix that commutes with the rotations which leave invariant one axis and permute cyclically all the others (with the appropriate sign) can only have non-zero elements on the diagonal. Thus, we need a different definition of 'isotropy', except for the matrices I and B. An intuitive formulation of the concept of isotropy is that, as one applies a sequence of factors of a particular matrix T to a given initial direction, then one explores equally all the different directions (i.e. sets of directions invariant under the action of T smaller than $2d$ are not formed). To be more precise, for the first $(d-1)$ applications of T we explore each direction (irrespective of the sense) exactly once. Otherwise, suppose one revisits on the nth application $(n < d-1)$ a direction already visited either in the same sense or the opposite sense. Then invariant sets of directions of size n or $2n$ (depending on the sense) are obtained. Therefore we wish to visit each direction in one sense only for the first $(d-1)$ applications of T. However we must then visit the other senses precisely once in the next d applications. Thus

$$T^d = -I \tag{1}$$

guaranteeing that one starts with the initial direction in the *opposite* sense after d applications. This new definition of isotropy implies that we move over all $2d$ directions after $(2d-1)$ applications of T.

A first consequence of (1) is that the trajectories in a sample, where all sites are of the type T, are all localised with $2d$ steps. This is because we take exactly two steps along each coordinate axis, in opposite directions, per cycle—causing no net motion. It is easy to see that if $T^d = +I$ then we get net motion.

A second consequence of (1) is that in odd dimensionality the determinant of T must be -1. However there is an ambiguity in the determinant for even dimensionalities. To resolve this question we will now give an algorithm for the construction and enumeration of the allowed T in an arbitrary dimension. The basis of the construction lies in the principles of reflection symmetry and avoiding invariant sets of directions of size less than d. Firstly, define the axis for the first incoming trajectory to be considered to be the positive x direction. Secondly, pick which direction (and sense) that trajectory exits on, say the α direction. There are $2(d-1)$ ways to choose this, the factor 2 coming from the senses of the axes. Use reflection symmetry to deduce that the incoming trajectory along the negative x direction must exit along the negative α direction. Now consider the trajectory incident in the α direction and pick an outgoing direction which cannot be along any axis considered so far, say β—there are $2(d-2)$ ways to do this. Use reflection symmetry, and so on. In the dth step we have no freedom since we must go to the negative x direction. The total number of ways to pick combinations of incoming and outgoing directions, and hence the number of allowed T matrices, $N(T)$, is

$$N(T)=(d-1)!2^{d-1}. \tag{2}$$

It is convenient to introduce the notation $T_{\alpha,\beta\ldots}$ for the T picked in the above manner. In figure 1 we show the relation between incoming and outgoing directions for the matrix $T_{-y,-x}$.

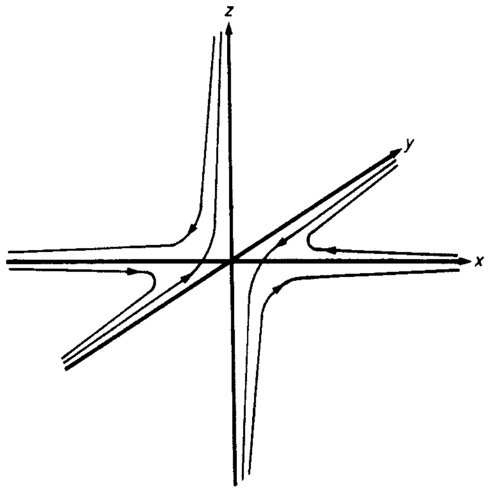

Figure 1. The relation of the incoming to the outgoing directions for the matrix $T_{-y,-z}$.

Let us express the determinant of T in terms of the conventional expansion—taking a sum of products of elements, one picked from each row and column. Then only one term in this sum contributes for a given T—as there is only one element in any row and column. Since the entries are all ± 1, the determinant is merely the product of all the (-1) elements in the matrix times the sign of the permutation associated with that contribution to the determinant. The latter is $(-1)^{d-1}$ as the permutation is a cycle of order d. To deduce the net sign of the product of the (-1) terms from the entries in T, consider letting T operate on a direction $(1, 0, 0, \ldots, 0)$. The result will be a vector which has zero in all components but one, which will be ± 1 depending on the sign of the non-zero element in the first row of T. Let T operate on this new vector, and then continue for d applications of T. The resultant vector must be $(\pm 1, 0, \ldots, 0)$ by the cyclic (by definition) nature of the matrix. The sign is merely the product of all the elements in T. However we insist that $T^d=-I$, so the product of all the signs must be (-1). Thus the determinant of T must be $(-1)^{d+1}(-1)=(-1)^d$. Thus in even dimensions the determinant of T is equal to $+1$.

Again using the fact that T only has one element per row or column and the fact that there are no invariant sets smaller than d, we may deduce that the secular equation for T is of the form:

$$(-\lambda)^d+\det(T)=0. \tag{3}$$

Thus $\lambda^d=-1$ or $\lambda_n=\exp(\mathrm{i}\pi/d)\exp(2\pi n\mathrm{i}/d)$ for $n=0, 1, \ldots, (d-1)$.

We also note in passing that not all the matrices T are linearly independent, as there are more than $d(d-1)/2$ of them. Also the matrices T, I and B do not in general form a group. The product of two T matrices must have an even number of (-1) elements and, therefore, cannot be another T matrix.

The parameter space for the model has a dimensionality equal to the number of distinct types of matrices, M, and by (2) is $(d-1)!2^{d-1}+2$. The allowed region of the space is determined by the plane $\Sigma P(M)=1$. We may obtain some bounds on the regions with extended trajectories by a trivial generalisation of the arguments used in two dimensions (Gunn and Ortuño 1985). Firstly, unless there is a connected cluster of non-backwards sites across the sample, i.e. $(1-P(B)) \geqslant p_c(d)$ (where $p_c(d)$ is the percolation probability for site percolation in d dimensions), there will be no extended trajectories. Note that roughly $p_c(d) \simeq 1/(2d)$—another indication that trajectories tend to be more delocalised in higher dimensions.

It is also clear that if there are only straight and backward sites then all trajectories are localised with the same average length as in two dimensions. We may extend the previous mean field calculation (moving away from this region). We will calculate the probability, σ, that, starting with a section of trajectory on a B site, the trajectory will not percolate across the sample. Percolation is equivalent to the trajectory being infinitely long, as trajectories have 'excluded volume' so they cannot merge. σ is the total probability of having a finite trajectory. The first possibility is to have a string of n I sites followed by a B. The sum of such probabilities is

$$\sigma_s = \frac{P(B)}{1-P(I)}. \tag{4}$$

The other possibility is a string of $n\,I$ followed by a T site which connects $\mu(d)$ non-percolating (but zero percolating) sections of trajectory. Thus the total probability of non-percolation is (after a little manipulation):

$$\sigma = \frac{\beta + \sigma^{\mu(d)}}{1+\beta} \tag{5}$$

where $\beta = P(B)/P(T)$. To see how percolation occurs let us define $\sigma = 1-\tau$ for $\tau \ll 1$. Thus τ is the probability of having an infinite trajectory. Then we find

$$\tau \approx \frac{2[(\mu((d)-1)-\beta]}{[\mu(d)(\mu(d)-1)]}. \tag{6}$$

We see that we get extended trajectories for $P(B) < (\mu(d)-1)P(T)$.

The value of $\mu(d)$ is determined as follows: consider what happens with sections of I plus a B connected as 'arms' onto such a T site. Note that the trajectory's directions can be described by a sequence of matrices $(\ldots B\,T\,B\,T\ldots)$ applied to the original direction. If the direction of the trajectory coincides with the original one after a product $(B\,T)^n$ (and not before) we may say that groups of n arms are connected by the T site. Since $T^d=-I$ and there are no invariant sets of directions of size less than d, we find that in odd dimensionalities d arms are connected (e.g. for $3d$ see figure 2) but that in even dimensionalities $2d$ arms are connected. Now in $\mu(d)$, we must omit the initial section of trajectory:

$$\mu(d) = [3+(-1)^d]d/2-1. \tag{7}$$

In conclusion, we have constructed a simple model for ballistic motion in a random environment, consisting of a lattice of quenched orthogonal matrices. Matrices were classified and studied which were 'isotropic' in a manner related to their cycle structure. A mean field theory of the transition between localised and extended trajectories revealed differences between even and odd dimensionalities.

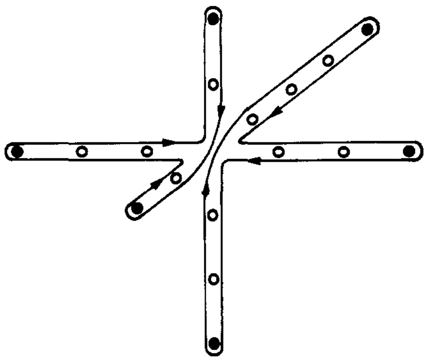

Figure 2. Sections of trajectory, ending in B sites (full circle), linked by a $T_{-1,-2}$ site. The open circles indicate I sites.

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